May 1, 2024 — Global Hecke Action (Notes & talk by Nikolay)

We specialize to Betti setting. This corresponds to setting (a) of GKRV. Thus we will cover material from the Toy model paper [GKRV] and Nadler–Yun paper.

In the end, we'll discuss the necessary modifications needed to upgrade to greater generality (Betti, deRham, etale). So NO "restricted" for now!

Let $k = \mathbb{C}$ and X = smooth projective curve$/k$.

Main Goal Construct an action (in Betti setting)

$$QCoh(LocSys_{G^\vee}(X)) \curvearrowright Shv^{all}_{Nilp}(Bun_G)$$

where • $QCoh(LocSys_{G^\vee}(X))(S) = \left\{ \begin{array}{l} \text{right } t\text{-exact } \otimes\text{-functors} \\ Rep\, G^\vee \longrightarrow LS(X) \otimes QCoh(X) \end{array} \right\}$

constant prestack, only remembers homotopy type of $X(\mathbb{C})$

and $LS(X) \subseteq Shv^{all}(X(\mathbb{C}))$ consists of complexes of all sheaves whose cohomology sheaves are locally constant (i.e. equivalent to any $\pi_1(X(\mathbb{C}))$-rep).

Note, $LS(X) = Shv_{lisse}(X)$ $(= Maps^{\otimes}(X, Vect))$

• $Shv^{all}_{Nilp}(Bun_G)$ = All sheaves (in $\mathbb{C}$-analytic topology) on Bun_G with singular support in $Nilp \subseteq T^*Bun_G$, in the sense of Kashiwara–Schapira (vanishing cycles vanish)

Outline: (I) Explain equivalence between

$$QCoh(LocSys_{G^\vee}(X)) \simeq (Rep\, G^\vee)^{\otimes X} := \text{"Chiral homology of } Rep\, G^\vee \text{ along } X\text{"}$$

(II) Construct a "Hecke functor" coming from geometric Satake

$$H : (Rep\, G^\vee)^{\otimes I} \otimes Shv^{all}(Bun_G) \longrightarrow Shv^{all}(Bun_G \times X^I).$$

(III) Show this action restricts to an action

$$H : (\mathrm{Rep}\, G^\vee)^{\otimes I} \otimes \mathrm{Shv}_{Nilp}(\mathrm{Bun}_G) \longrightarrow \mathrm{Shv}_{Nilp \times \{0\}}(\mathrm{Bun}_G \times X^I)$$

following an argument by [Nadler-Yun], and then conclude by using

$$\mathrm{Shv}_{Nilp \times \{0\}}(\mathrm{Bun}_{G \times X^I}) = \mathrm{Shv}_{Nilp}(\mathrm{Bun}_G) \otimes LS(X).$$

(I) Generalities of categorical actions:

- Let $\mathcal{A}$ = symmetric monoidal dg category. we'll work with $\mathcal{A} = \mathrm{Rep}\, G^\vee$, or more generally $QCoh(Z)$ for a nice stack Z.
- Let $DGCat^{SymMon}$ denote the category of all symmetric monoidal dg categories.
- Let X be thought of as an object of Spc by taking its homotopy type.

Def $\mathcal{A}^{\otimes X} := \operatorname{colim}_X \mathcal{A} \in DGCat^{SymMon}$ "chiral homology of $\mathcal{A}$ along X." denotes the colimit along index category X of the constant functor $X \longrightarrow DGCat^{SymMon}$ with value $\mathcal{A}$.

It is characterized by the universal property

$$\underbrace{\mathrm{Maps}^{\otimes}}_{\text{in } DGCat^{SymMon}}(\mathcal{A}^{\otimes X}, \mathcal{B}) = \mathrm{Maps}_{Spc}(X, \mathrm{Maps}^{\otimes}(\mathcal{A}, \mathcal{B}))$$

Remark: Imposing Maps preserve $\otimes$ (instead of just in DGCat) is strong. Removing this restriction gives $\mathcal{A}^X \in DGCat$, characterized by universal property:

$$\mathrm{Map}_{\mathrm{DGCat}}(\mathcal{A}^{X}, \mathcal{B}) = \mathrm{Map}_{\mathrm{Spc}}(X, \mathrm{Map}(\mathcal{A},\mathcal{B})) \simeq \mathrm{Map}_{\mathrm{DGCat}}(LS(X), \mathrm{Map}(\mathcal{A},\mathcal{B}))$$
$$\simeq \mathrm{Map}_{\mathrm{DGCat}}(LS(X)\otimes\mathcal{A}, \mathcal{B})$$
$$\Longrightarrow \boxed{\mathcal{A}^{X} = LS(X)\otimes\mathcal{A}}, \quad LS(X) = \mathrm{Maps}(X, \mathrm{Vect})$$

It's immediate that for $X = I$, a finite set,
$$\mathcal{A}^{\otimes I} = \mathcal{A}\otimes\cdots\otimes\mathcal{A} \quad |I|\text{-many times.}$$
In general,
$$\mathcal{A}^{\otimes X} = \text{left Kan extension (LKE) of } \left(\begin{array}{c} \mathrm{fSet} \longrightarrow \mathrm{DGCat}^{\mathrm{SymMon}} \\ I \longmapsto \mathcal{A}^{\otimes I} \end{array}\right)$$
along

$$\begin{tikzcd} \mathrm{fSet} \arrow[r, hook] \arrow[dr, "I"'] & \mathrm{Spc} \arrow[d, dashed, "\mathrm{LKE}"] \\ & \mathcal{A}^{\otimes I} \end{tikzcd}$$

Rmk $\mathcal{A}^{\otimes X}$ preserves all colimits in X by universal property.

$$\boxed{\begin{array}{l} \underline{\text{Main Thm}} \quad \text{Let } X = \text{smooth projective curve}/k. \text{ Then} \\ \qquad (\mathrm{Rep}\, G^{\vee})^{\otimes X} \simeq \mathrm{QCoh}(\mathrm{LocSys}_{G^{\vee}}(X)) \end{array}}$$

Pf Sketch It's easier to prove a more general statement:

Suppose Z is a "nice stack": Z is affine, and

(1) $Z \xrightarrow{\Delta} Z\times Z$ is affine, and

(2) $\mathrm{QCoh}(Z)$ is dualizable.

Then we'll prove: (Recover Thm by $Z = pt/G^{\vee}$)

$\underline{\text{Thm}'}$ $$\mathrm{QCoh}(Z)^{\otimes X} \xrightarrow{\sim} \mathrm{QCoh}(\underline{\mathrm{Maps}}(X,Z)).$$

Pf Write $Z^{X} := \underline{\mathrm{Maps}}(X, Z)$. We have maps

$$X = \mathrm{Map}_{\mathrm{Spc}}(pt, X) \xrightarrow{\mathrm{Map}(-,Z)} \mathrm{Map}_{\mathrm{PreStk}}(\underline{\mathrm{Map}}(X,Z), Z)$$
$$\xrightarrow{\mathrm{QCoh}} \mathrm{Map}_{\mathrm{DGCat}^{\mathrm{SymMon}}}(\mathrm{QCoh}(Z), \mathrm{QCoh}(\underline{\mathrm{Map}}(X,Z))).$$

Thus, universal property implies we have a map:

$$(QCoh(Z))^{\otimes X} \longrightarrow QCoh(Z^X).$$

Remark: if X is finite set, it's obvious this is isomorphism by $\otimes$-product decomp of QCoh (just need $QCoh(Z)$ dualizable).

Now there are three ingredients:

<u>Step 1</u> We may write $X = \underset{i \in I}{\operatorname{colim}} \bigvee_{I_i} S^1$, for a $\overbrace{\text{sifted}}^{\text{preserves filtered colim + geom. realization}}$ colimit, since X is a connected space over $\mathbb{C}$. (Classical Alg. Top Thm!)

<u>Step 2</u> For $X = \bigvee_I S^1$, prove $QCoh(Z^{\vee_I S^1}) = QCoh(Z) \underset{QCoh(Z^{I_+})}{\otimes} QCoh(Z)$

$$\begin{tikzcd} \overbrace{I \sqcup pt}^{I_+} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & pt \arrow[d] \\ pt \arrow[r] & \bigvee_I S^1 \end{tikzcd} \rightsquigarrow \begin{tikzcd} QCoh(Z^{I_+}) & QCoh(Z) \arrow[l] \\ QCoh(Z) \arrow[u] & QCoh(Z^{\vee S^1}) \arrow[l] \arrow[u] \arrow[ul, phantom, "\lrcorner"] \end{tikzcd}$$

This follows formally from Z being a "nice stack."

<u>Step 3</u> $QCoh(Z^X) = \underset{i \in I}{\operatorname{colim}} \underline{QCoh(Z^{\vee_{I_i} S^1})}$,

w/ colimit in DGCat, (b/c I sifted!).

Again, this formally follows from Z being nice.

To finish:

$$QCoh(Z)^{\otimes X} \overset{\text{Universal property}}{=\!=} \underset{i \in I}{\operatorname{colim}} QCoh(Z)^{\otimes \vee_{I_i} S^1} \overset{\text{step 2}}{=\!=} \underset{i \in I}{\operatorname{colim}} QCoh(Z^{\vee_{I_i} S^1}) \overset{\text{step 3}}{=\!=} QCoh(Z^X) \quad \blacksquare$$

As a corollary of Main Thm,

Giving a (categorical) action

$$QCoh(LocSys_{G^\vee}(X)) \curvearrowright \mathcal{M} = \text{any dualizable dg monoidal cat, e.g. } Shv(Bun_G),$$

$\overset{\text{Thm}}{\Longleftrightarrow}$ Giving an action

$\mathcal{M}$ dualizable, monoidal $(\mathrm{Rep}\, G^\vee)^{\otimes X} \curvearrowright \mathcal{M}$

$\Longleftrightarrow$ for any finite set I, giving a system of monoidal functors

$\mathrm{Rep}(\underbrace{G^\vee \times \cdots \times G^\vee}_{|I|}) = (\mathrm{Rep}\, G^\vee)^{\otimes I} \longrightarrow \mathrm{End}(\mathcal{M}) \otimes LS(X^I)$

Plus compatibility: for any $I \longrightarrow J$, a commutative diagram

$$\begin{tikzcd}
(\mathrm{Rep}\, G^\vee)^{\otimes I} \arrow[r] \arrow[d] \arrow[dr, phantom, "\curvearrowright"] & \mathrm{End}(\mathcal{M}) \otimes LS(X^I) \arrow[d] \\
\mathrm{Rep}(G^\vee)^{\otimes J} \arrow[r] & \mathrm{End}(\mathcal{M}) \otimes LS(X^J)
\end{tikzcd}$$

Plus higher algebra compatibilities $\blacksquare$

Break

(II) Geometric Satake Functor

Recall the Hecke stacks:

Def $\mathrm{Hecke}_I(S) := \left\{ (x^I, \mathcal{P}', \mathcal{P}'', \alpha) : \begin{array}{l} \cdot\ x^I \text{ is } I\text{-tuple of } S\text{-points of } X \\ \cdot\ \mathcal{P}', \mathcal{P}'' \in \mathrm{Bun}_G(X)(S) \\ \cdot\ \alpha : \mathcal{P}'|_{S\times X \setminus \Gamma_{x^I}} \xrightarrow{\sim} \mathcal{P}''|_{S\times X\setminus \Gamma_{x^I}} \end{array} \right\}$

$\mathrm{Hecke}^{loc}_I(S) := \left\{ (x^I, \mathcal{P}', \mathcal{P}'', \alpha) : \begin{array}{l} \cdot\ x^I \text{ is } I\text{-tuple of } S\text{-points of } X \\ \cdot\ \mathcal{P}', \mathcal{P}'' \text{ are } G\text{-bundles on } D_{x^I} \\ \text{(technically, completion of } S\times X \text{ along } \Gamma_{x^I}\text{.)} \\ \cdot\ \alpha : \mathcal{P}'|_{\mathring{D}_{x^I}} \xrightarrow{\sim} \mathcal{P}''|_{\mathring{D}_{x^I}} \end{array} \right\}$

Also recall, uniformization thm produces an isom of stacks

$$\mathrm{Hecke}^{loc}_{I=pt} = G(\mathcal{O}_x)\backslash G(K_x)/G(\mathcal{O}_x)$$

Thm (Geometric Satake) There exists a natural monoidal functor (which is equivalence for $I = pt$):

$$\Phi_I : (\mathrm{Rep}\, G^\vee{}^{\otimes I})^\heartsuit \longrightarrow \mathrm{Shv}(\mathrm{Hecke}^{loc}_I)^\heartsuit := \mathrm{Shv}(\mathrm{Gr}_{G,I})^{(L^+G)_I}$$

where RHS consists of $G(\mathcal{O})$-equivariant perverse sheaves on the affine Grassmannian, and is called the Satake category.

- By t-exactness of Φ_I, it extends to a functor

$$S : (\mathrm{Rep}\, G^\vee)^{\otimes I} \longrightarrow \mathrm{Shv}(\mathrm{Hecke}^{loc}_I),$$

which we call the Satake functor.

We have the following correspondence diagram:

$$\begin{tikzcd} & \mathrm{Hecke}_I \arrow[dl, "\overleftarrow{h}\times r"', "\text{smooth}"] \arrow[dr, "\overrightarrow{h}\times s", "\text{proper}"'] & \\ \mathrm{Bun}_G \times \mathrm{Hecke}_I^{loc} & & \mathrm{Bun}_G \times X^I \end{tikzcd}$$

where r = restrict to $D_x \to S \times X$

and $s = (\mathrm{Hecke} \xrightarrow{r} \mathrm{Hecke}_x^{loc} \xrightarrow{\pi} X)$

Finally, we may construct the Hecke functor:

$$\begin{aligned} H_V : \mathrm{Shv}(\mathrm{Bun}_G) &\longrightarrow \mathrm{Shv}(\mathrm{Bun}_G \times X^I) \\ \mathcal{F} &\longmapsto (\overrightarrow{h} \times s)_* (\overleftarrow{h} \times r)^! (\mathcal{F} \boxtimes \mathcal{S}_V) \end{aligned}$$

(III) Restricting Hecke Functor to $\mathrm{Shv}_{Nilp}(\mathrm{Bun}_G)$.

Our main goal is to prove the following theorem typically attributed to Nadler & Yun: (Remark: Ginzburg in fact proved this statement in §6.6 of his paper on "Perverse sheaves on a loop group & Langlands duality"!)

Thm The Hecke functor restricts to give a functor

$$H_V : \mathrm{Shv}_{Nilp}(\mathrm{Bun}_G) \longrightarrow \mathrm{Shv}_{Nilp \times \{0\}}(\mathrm{Bun}_G \times X^I)$$

Pf By associativity of $H : (\mathrm{Rep}\, G^\vee)^{\otimes I} \otimes \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Bun}_G \times X^I)$, we may assume $I = pt$.

Next, recall the two characteristics of singular support with respect to smooth pullback $(\overleftarrow{h} \times r)$ & proper pushforward $(\overrightarrow{h} \times s)$:

Given $F : U \longrightarrow V$, consider

$$\begin{tikzcd} T^*U & T^*V \times_V U \arrow[l, "dF"'] \arrow[r, "f_\natural"] & T^*V \arrow[from=1-1, to=1-3, bend left, "(dF)_*"] \end{tikzcd}$$

i.e. $(dF)_* := f_\natural \circ dF^{-1}$

$(dF)^* := dF \circ f_\natural^{-1}$

$$\overset{\text{WANT } \mathcal{N}\times\{0\}}{SS(H_V(\mathcal{F})) \subseteq d(\vec{h}\times s)_*\, d(\overleftarrow{h}\times r)^*\big(SS(\mathcal{F}\boxtimes S_V)\big) \subseteq T^*(\mathrm{Bun}_G\times X)}$$

$(df)^*$

GOAL!

Now, consider $(\xi', \xi_H) \in SS(\mathcal{F}\boxtimes S_V) = \mathcal{N} \times SS(S_V)$

and $(\xi'', \xi_X) \in SS(H_V(\mathcal{F}))$ such that

$$(\dagger) \qquad d(\overleftarrow{h}\times r)^*(\xi', \xi_H) = d(\vec{h}\times s)^*(\xi'', \xi_X).$$

WANT TO SHOW ξ'' also nilpotent and $\xi_X = 0$.

Lets consider the fiber over $x\in X$ of the differential of the Hecke correspondence:

$$T^*_{(\mathcal{P}',\mathcal{P}'',\varphi)}(\mathrm{Hecke}_x) = \big[(\mathfrak{g}^*_{\mathcal{P}'}+\mathfrak{g}^*_{\mathcal{P}''})\otimes\omega_X \text{ plus compatibility}\big]$$

$$\begin{tikzcd}
 & T^*_{(\mathcal{P}',\mathcal{P}'',\varphi)}(\mathrm{Hecke}_x) & \\
\underbrace{T^*_{\mathcal{P}'}(\mathrm{Bun}_G)}_{\mathfrak{g}^*_{\mathcal{P}'}\otimes\omega_X} \times \underbrace{T^*_{(\mathcal{P}',\mathcal{P}'',\varphi)|_{D_x}}(\mathrm{Hecke}^{loc}_x)}_{(\mathfrak{g}^*_{\mathcal{P}'}+\mathfrak{g}^*_{\mathcal{P}''})\otimes\omega_X|_{D_x} + \text{compatibility}} \arrow[ur, "d(\overleftarrow{h}_x\times r_x)^*"] & & T^*_{\mathcal{P}''}(\mathrm{Bun}_G) \arrow[ul, "d(\vec{h}_x)^*"']
\end{tikzcd}$$

$(s_x : \mathrm{Hecke}_x \to \{x\},\ ds_x = 0)$

$T^*_{\mathcal{P}''}(\mathrm{Bun}_G) = \mathfrak{g}^*_{\mathcal{P}''}\otimes\omega_X$

Recall:

$$T^*_{\mathcal{P}'}(\mathrm{Bun}_G) = H^1(X, \mathfrak{g}_{\mathcal{P}'})^* = \Gamma(X, \mathfrak{g}^*_{\mathcal{P}'}\otimes\omega_X).$$

$$T^*_{(\mathcal{P}',\mathcal{P}'',\varphi)}(\mathrm{Hecke}_x) = H^1(X, K)^*, \text{ where}$$

$$K = \mathrm{Cone}\big(\mathfrak{g}_{\mathcal{P}'}\oplus\mathfrak{g}_{\mathcal{P}''} \longrightarrow j_*j^*\mathfrak{g}_{\mathcal{P}'} \simeq j_*j^*\mathfrak{g}_{\mathcal{P}''}\big),\ j : X\setminus x \hookrightarrow X.$$

$$\begin{tikzcd}
T^*_{(\mathcal{P}',\mathcal{P}'',\varphi)}(\mathrm{Hecke}_x) \\
T^*_{(\mathcal{P}',\mathcal{P}'',\varphi)|_{D_x}}(\mathrm{Hecke}^{loc}_x) \arrow[u, "dr_x^*"']
\end{tikzcd}$$

$$T^*_{(\mathcal{P}',\mathcal{P}'',\varphi)|_{D_x}}(\mathrm{Hecke}^{loc}_x) = H^1(D_x, K)^*$$

$$(\dagger\dagger) \qquad \overset{\text{explicitly}}{=} \left\{ \begin{array}{l} (\xi'_{loc}, \xi''_{loc}) \in \Gamma(X, (\mathfrak{g}^*_{\mathcal{P}'}\oplus\mathfrak{g}^*_{\mathcal{P}''})\otimes\omega_X) \text{ such that} \\ \varphi(\xi'_{loc}|_{D_x^\circ}) = \xi''_{loc}|_{D_x^\circ} \end{array} \right\}$$

Under these identifications, $d\overleftarrow{h}_x^*$, $d\overrightarrow{h}_x^*$ are just induced by the inclusions

$$\begin{tikzcd} & \mathfrak{g}_{\rho'}^* + \mathfrak{g}_{\rho''}^* & \\ \mathfrak{g}_{\rho'}^* \arrow[ur, hook] & & \mathfrak{g}_{\rho''}^* \arrow[ul, hook'] \end{tikzcd}.$$

The fiber of (†) over $x \in X \Rightarrow$

$$d\overleftarrow{h}_x^*(\xi') + (dr_x)^*(\underbrace{\xi_{H_x}}_{\in T^*(\mathrm{Hecke}_x^{loc})}) = d\overrightarrow{h}_x^*(\xi'') \in T^*(\mathrm{Hecke}_x)$$

Thus ξ' nilpotent $\Leftrightarrow \xi'|_{D_x^o}$ nilp.

$\Leftrightarrow \xi''|_{D_x^o}$ nilp

$\Leftrightarrow \xi''$ nilp.

The compatibility constraint of covectors on Hecke stack ensures one covector is nilp $\Leftrightarrow$ the other is nilp

Remains to show $\xi_X = 0$.

Sketch: The SES admits a canonical splitting:

$$0 \to T_x^* X \to T^*_{(-)}(\mathrm{Hecke}_X) \to T(\mathrm{Hecke}) \to 0$$

ξ_H

Thus, suffices: If $\xi_H \in T^*(\mathrm{Hecke}_X)$ is in $SS(S_V)$ & $\xi_{H_x} = (\xi'_{loc}, \xi''_{loc}) \in T^*(\mathrm{Hecke}_x^{loc})$ is nilpotent, then image of ξ_H under splitting is 0.

This is done by an explicit calculation, by working with $X = \mathbb{A}^1$, $x = 0$ (since statement is local), using

$(\rho', \rho'', \alpha) \in \mathrm{Hecke}_0^{loc} = G(O)\backslash G(K)/G(O) \rightsquigarrow g \in G(K)$ + data.

Then, use Cartan decomp to reduce to $g = t^\lambda$, where $\mathcal{g}_K = \mathbb{C}((t))$.

Then, the canonical splitting takes the form:

$$T^*\mathrm{Hecke}_0^{loc} = \mathfrak{g}_0^* dt \cap \mathrm{Ad}_{t^\lambda}(\mathfrak{g}_0^*)\, dt \subseteq \mathfrak{g}_K^*\, dt$$

$$\begin{tikzcd} T^*\mathrm{Hecke}_0^{loc} \arrow[d, "\text{splitting}"'] & \xi \arrow[d, mapsto] \\ T_0^*\mathbb{A}^1 = \mathbb{C} & \mathrm{Res}_{t=0}\langle \underbrace{\lambda t^{-1}}_{d\log(g)}, \xi \rangle \end{tikzcd}$$

Reduced to showing: If $\bar{\xi}_H := \xi_H \bmod dt \in \mathfrak{g}^*$ is nilpotent, then $\langle \lambda, \bar{\xi}_H \rangle$ $(= \mathrm{Res}_{t=0}\langle \lambda t^{-1}, \xi_H \rangle)$ is zero. Exercise! ■

Finally, the last ingredient we need is the following:

Suppose $\mathcal{N}, \mathcal{N}'$ are Lagrangian substacks of T^*X, T^*X'. Then
$$\mathrm{Shv}_{\mathcal{N}\times\mathcal{N}'}(X \times Y) = \mathrm{Shv}_{\mathcal{N}}(X) \otimes \mathrm{Shv}_{\mathcal{N}'}(Y)$$
(this kind of generality only holds in Betti setting).

(IV) Construction of the global Hecke action in general

(Betti, de Rham, etale).

Let us explain the construction in general

First, we must replace $\mathrm{LocSys}_{G^\vee}(X)$ with $\mathrm{LocSys}_{G^\vee}^{restr}(X)$,

where, recall from Nick's talk,

$$\mathrm{LocSys}_{G^\vee}^{restr}(X)(S) = \left\{ \begin{array}{c} \text{right } t\text{-exact } \otimes\text{-functors} \\ \mathrm{Rep}\, G^\vee \longrightarrow \mathrm{QLisse}(X) \otimes \mathrm{QCoh}(S) \end{array} \right\}, \text{ where}$$

- $\mathrm{QLisse}^{Betti}(X) \subseteq \mathrm{Shv}_{l.c.}(\underline{X(\mathbb{C})})$ is complexes whose cohom. is given by ind-fin.dim. $\pi_1(X(\mathbb{C}))$-reps,
- $\mathrm{QLisse}^{et}(X) \subseteq \mathrm{Shv}^{et}(X)$ consists of complexes of Ind-constructible l-adic sheaves w/ ind-locally constant cohomology sheaves,

• $QLisse^{dR}(X) \subseteq D\text{-mod}(X)$ consists of ind-(regular local systems).

Main Thm (8.3.7 AGKRRV) X = smooth, proper curve.

Then $(\mathrm{Rep}\,G^\vee)^{\otimes X\text{-lisse}} := \underline{\mathrm{coHom}}(\mathrm{Rep}\,G^\vee, QLisse(X))$

is equivalent to $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{G^\vee})$.

Remark (1) Given $Q \in \mathrm{DGCat}^{\mathrm{SymMon}}$ which is dualizable, & $\mathcal{C}, \mathcal{D} \in \mathrm{DGCat}^{\mathrm{SymMon}}$, define $\mathrm{coHom}(\mathcal{C}, Q) \in \mathrm{DGCat}^{\mathrm{SymMon}}$ using universal property:

$\mathrm{Fun}^{\otimes}(\underline{\mathrm{coHom}}(\mathcal{C}, Q), \mathcal{D}) := \mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D} \otimes Q)$.

(2) $\underline{\mathrm{coHom}}(\mathrm{Rep}\,G^\vee, LS(X)) = (\mathrm{Rep}\,G^\vee)^{\otimes X}$.

(3) Even in Betti setting we proved, additional work is needed to go from Betti to restricted Betti.

(Note, we're now working with Ind-constructible sheaves instead of all sheaves!).

Main Thm 2 [Nadler-Yun analogue] There is an action

$$(\mathrm{Rep}\,G^\vee)^{\otimes I} \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \longrightarrow \mathrm{Shv}_{\mathrm{Nilp} \times \{0\}}(\mathrm{Bun}_G \times X^I)$$

where $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is interpreted differently, depending on the sheaf theory. E.g., for etale, we use Sasha's definition.

The final ingredient is the decomposition

$$\mathrm{Shv}_{\mathrm{Nilp} \times \{0\}}(\mathrm{Bun}_G \times X^I) = \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes QLisse(X)^{\otimes I}.$$

Cor There exists an action $QCoh(LocSys^{rest}_{G^\vee}) \curvearrowright Shv_{Nilp}(Bun_G)$