

May 1, 2024

Global Hodge Action

(Notes & talk by Nikolaj)

We specialize to Betti setting. This corresponds to setting (a) of GKRV. Thus we will cover material from the

Toy model paper [GKRV] and Nadler-Yun paper.

In the end, we'll discuss the necessary modifications needed to upgrade to greater generality (Betti, de Rham, etc). So NO "restricted" for now!

Let $k = \mathbb{C}$ and $X =$ smooth projective curve / k .

Main Goal Construct an action (in Betti setting)

$$QCoh(\text{LocSys}_{G^v}(X)) \rightsquigarrow \text{Shv}_{\text{Nilp}}^{\text{all}}(\text{Bun}_G)$$

where $\bullet QCoh(\text{LocSys}_{G^v}(X))(S) = \left\{ \begin{array}{l} \text{right t-exact } \otimes\text{-functors} \\ \text{Rep } G^v \rightarrow LS(X) \otimes QCoh(X) \end{array} \right\}$

and $LS(X) \cong \text{Shv}^{\text{all}}(X(\mathbb{C}))$ consists of complexes of all sheaves whose cohomology sheaves are locally constant (ie equivalent to our $\pi_0(X(\mathbb{C}))$ -rep).
 constant prestack, only remembers homotopy type of $X(\mathbb{C})$

Note, $LS(X) = \text{Shv}_{\text{lisse}}(X) (= \text{Maps}^{\otimes}(X, \text{Vect}))$

- $\bullet \text{Shv}_{\text{Nilp}}^{\text{all}}(\text{Bun}_G) =$ All sheaves (in \mathbb{C} -analytic topology) on Bun_G with singular support in $\text{Nilp} \subseteq T^* \text{Bun}_G$, in the sense of Kashiwara-Schapira (vanishing cycles vanish)

Outline: (I) Explain equivalence between

$$QCoh(\text{LocSys}_{G^v}(X)) \cong (\text{Rep } G^v)^{\otimes X} := \text{"Chiral homology of Rep } G^v \text{ along } X\text{"}$$

(II) Construct a "Hecke functor" coming from geometric

$$\text{Satake } H : (\text{Rep } G^v)^{\otimes I} \otimes \text{Shv}^{\text{all}}(\text{Bun}_G) \rightarrow \text{Shv}^{\text{all}}(\text{Bun}_G \times X^I).$$

(III) Show this action restricts to an action

$$H : (\text{Rep } G^V)^{\otimes \mathbb{I}} \otimes \text{Shv}_{N/p}(\text{Bun}_G) \rightarrow \text{Shv}_{N/p \times \mathbb{I}}(\text{Bun}_{G \times X^{\mathbb{I}}})$$

following an argument by [Nader-Yun], and then conclude by using

$$\text{Shv}_{N/p \times \mathbb{I}}(\text{Bun}_{G \times X^{\mathbb{I}}}) = \text{Shv}_{N/p}(\text{Bun}_G) \otimes \text{LS}(X).$$

(I) Generalities of categorical actions:

- Let \mathcal{A} = symmetric monoidal dg category
we'll work with $\mathcal{A} = \text{Rep } G^V$, or more generally $\text{QCoh}(Z)$
for a nice stack Z .

- Let $\text{DGCat}^{\text{Sym Mon}}$ denote the category of all symmetric monoidal dg categories.
- Let X be thought of as an object of Spc by taking its homology type.

Def $\mathcal{A}^{\otimes X} := \text{colim}_X \mathcal{A} \in \text{DGCat}^{\text{Sym Mon}}$ "Chiral homology of \mathcal{A} along X ."

denotes the colimit along index category X of the constant functor $X \rightarrow \text{DGCat}^{\text{Sym Mon}}$ with value \mathcal{A} .

It is characterized by the universal property

$$\text{Maps}_{\text{DGCat}^{\text{Sym Mon}}}(\mathcal{A}^{\otimes X}, \mathcal{B}) = \text{Maps}_{\text{Spc}}(X, \text{Maps}_{\text{DGCat}^{\text{Sym Mon}}}(\mathcal{A}, \mathcal{B}))$$

Remark: Imposing Maps preserve \otimes (instead of just in DGCat) is strong. Removing this restriction gives $\mathcal{A}^X \in \text{DGCat}$, characterized by universal property:

$$\text{Map}_{\text{DGcat}}(A^X, B) = \text{Map}_{\text{Spc}}(X, \text{Map}(A, B)) \simeq \text{Map}_{\text{DGcat}}(\text{LS}(X), \text{Map}(A, B))$$

$$\simeq \text{Map}_{\text{DGcat}}(\text{LS}(X) \otimes A, B)$$

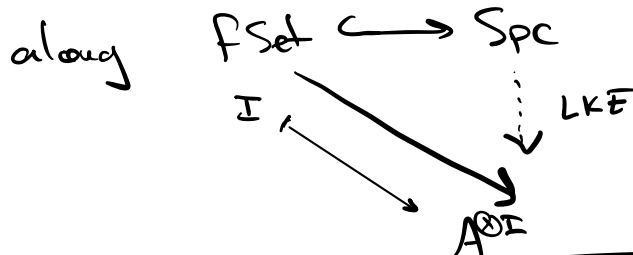
$$\Rightarrow \boxed{A^X = \text{LS}(X) \otimes A}, \quad \text{LS}(X) = \text{Maps}(X, \text{Vect})$$

It's immediate that for $X = I$, a finite set,

$$A^{\otimes I} = A \otimes \dots \otimes A \quad |I| \text{-many times. In general,}$$

$$A^{\otimes X} \simeq \text{left Kan extension of } (F\text{Set} \rightarrow \text{DGcat}^{\text{Sym Mon}})$$

$$I \longmapsto A^{\otimes I}$$



Rank $A^{\otimes X}$ preserves all colimits in X by universal property.

Main Thm Let $X = \text{smooth projective curve}/k$. Then

$$(\text{Rep } G^v)^{\otimes X} \simeq \text{QCoh}(\text{Loc Sys}_{G^v}(X))$$

PF Sketch It's easier to prove a more general statement:

Suppose Z is a "nice stack":

- (1) $Z \rightrightarrows Z \times Z$ is affine, and
- (2) $\text{QCoh}(Z)$ is dualizable.

Then we'll prove:

(Recover Thm by $Z = \text{pt}/G^v$)

$$\text{Thm}' \quad \text{QCoh}(Z)^{\otimes X} \simeq \text{QCoh}(\text{Maps}(X, Z)).$$

PF Write $Z^X := \text{Maps}(X, Z)$. We have maps

$$X = \text{Map}_{\text{Spc}}(\text{pt}, X) \xrightarrow{\text{Map}(-, Z)} \text{Map}_{\text{prestk}}(\text{Map}(X, Z), Z)$$

$$\xrightarrow{\text{QCoh}} \text{Map}_{\text{DGcat}^{\text{Sym Mon}}}(\text{QCoh}(Z), \text{QCoh}(\text{Map}(X, Z)))$$

Thus, universal property implies we have a map:

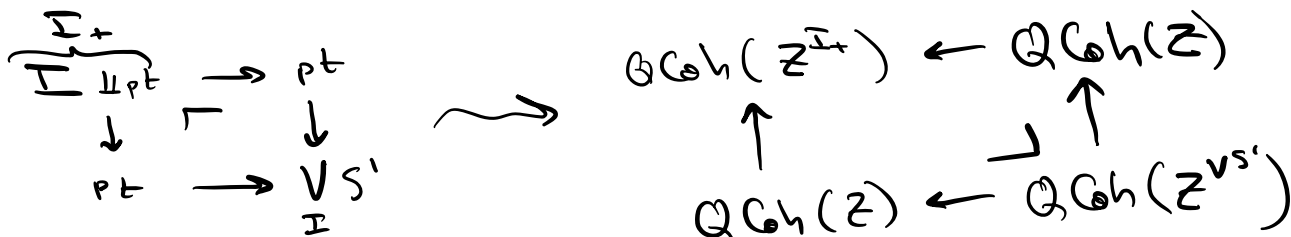
$$(\mathcal{Q}\text{Coh}(Z))^{\otimes X} \longrightarrow \mathcal{Q}\text{Coh}(Z^X).$$

Remark: if X is finite set, it's obvious this is isomorphism by \otimes -product decomp of $\mathcal{Q}\text{Coh}$ (just need $\mathcal{Q}\text{Coh}(Z)$ dualizable).

Now there are three ingredients:

Step 1 We may write $X = \text{colim}_{i \in I} \bigvee_{I_i} S^i$, for a sifted colimit, since X is a connected space over \mathbb{Q} . (Classical Alg. Top Thm!) preserves filtered colim + geom. realiz.

Step 2 For $X = \bigvee_I S^i$, prove $\mathcal{Q}\text{Coh}(Z^{\bigvee_I S^i}) = \mathcal{Q}\text{Coh}(Z) \otimes_{\mathcal{Q}\text{Coh}(Z^{I_+})} \mathcal{Q}\text{Coh}(Z)$



This follows formally, from Z being a "nice stack."

Step 3 $\mathcal{Q}\text{Coh}(Z^X) = \text{colim}_{i \in I} \mathcal{Q}\text{Coh}(Z^{\bigvee_{I_i} S^i})$

w/ colimit in $\mathcal{D}\mathcal{G}\text{Cat}$, (b/c I sifted!).

Again, this formally follows from Z being nice.

To finish:

$$\mathcal{Q}\text{Coh}(Z)^{\otimes X} \xrightarrow[\text{universal property}]{\text{universal}} \text{colim}_{i \in I} \mathcal{Q}\text{Coh}(Z)^{\otimes \bigvee_{I_i} S^i} \xrightarrow{\text{step 2}} \text{colim}_{i \in I} \mathcal{Q}\text{Coh}(Z^{\bigvee_{I_i} S^i}) \xrightarrow{\text{step 3}} \mathcal{Q}\text{Coh}(Z^X)$$

As a corollary of Main Thm,

Giving a (categorical) action

$$\mathcal{Q}\text{Coh}(\text{Loc Sys}_{\text{Gr}}(X)) \rightsquigarrow \mathcal{M} = \text{any } \text{dualizable dg monoidal cat, e.g. } \text{Shv}(\text{Bun}_G)$$

Thm \iff Giving an action

\mathcal{M} dualizable monoidal $(\text{Rep } G^v)^{\otimes X} \rightarrow \mathcal{M}$

\iff For any finite set I , giving a system of monoidal functors

$\text{Rep}(G^v \times_{|I|} G^v) = (\text{Rep } G^v)^{\otimes I} \rightarrow \text{End}(\mathcal{M}) \otimes \text{LS}(X^I)$

Plus compatibility: for any $I \rightarrow J$, a commutative diagram

$$\begin{array}{ccc} (\text{Rep } G^v)^{\otimes I} & \rightarrow & \text{End}(\mathcal{M}) \otimes \text{LS}(X^I) \\ \downarrow & \curvearrowright & \downarrow \\ (\text{Rep } G^v)^{\otimes J} & \rightarrow & \text{End}(\mathcal{M}) \otimes \text{LS}(X^J) \end{array}$$

Plus higher algebra compatibility ▣

Break

(II) Geometric Satake Functor

Recall the Hecke stacks:

Def $\text{Hecke}_{\mathbf{I}}(S) := \left\{ (x^{\mathbf{I}}, \mathcal{P}', \mathcal{P}'', \alpha) : \begin{array}{l} \cdot x^{\mathbf{I}} \text{ is } \mathbf{I}\text{-tuple of } S\text{-points of } X \\ \cdot \mathcal{P}', \mathcal{P}'' \in \text{Bun}_G(X)(S) \\ \cdot \alpha : \mathcal{P}'|_{S \times X / \Gamma_{x^{\mathbf{I}}}} \xrightarrow{\sim} \mathcal{P}''|_{S \times X / \Gamma_{x^{\mathbf{I}}}'} \end{array} \right\}$

$\text{Hecke}_{\mathbf{I}}^{\text{loc}}(S) := \left\{ (x^{\mathbf{I}}, \mathcal{P}', \mathcal{P}'', \alpha) : \begin{array}{l} \cdot x^{\mathbf{I}} \text{ is } \mathbf{I}\text{-tuple of } S\text{-points of } X \\ \cdot \mathcal{P}', \mathcal{P}'' \text{ are } G\text{-bundles on } D_{x^{\mathbf{I}}} \\ \text{(technically, completion of } S \times X \text{ along } \Gamma_{x^{\mathbf{I}}}) \\ \cdot \alpha : \mathcal{P}'|_{D_{x^{\mathbf{I}}}} \xrightarrow{\sim} \mathcal{P}''|_{D_{x^{\mathbf{I}}}} \end{array} \right\}$

Also recall, uniformization thus produces an isom of stacks

$$\text{Hecke}_{\mathbf{I}=\text{pt}}^{\text{loc}} = G(\mathcal{O}_x) \backslash G(K_x) / G(\mathcal{O}_x)$$

Thm (Geometric Satake) There exists a natural monoidal functor (which is equivalence for $\mathbf{I} = \text{pt}$):

$$\overline{\mathcal{D}}_{\mathbf{I}} : (\text{Rep } G^{\vee})^{\otimes \mathbf{I}} \longrightarrow \text{Shv}(\text{Hecke}_{\mathbf{I}}^{\text{loc}}) := \text{Shv}(Gr_{G, \mathbf{I}})^{(\mathbb{Z}^+)_G}_{\mathbf{I}}$$

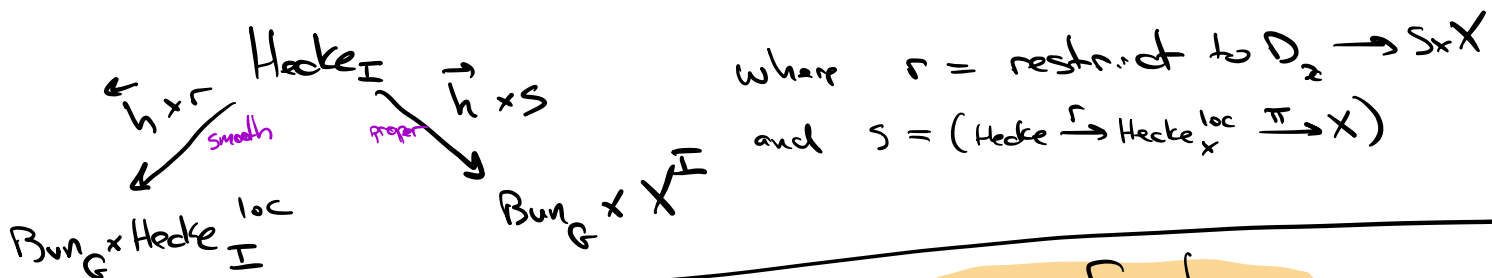
where RHS consists of $G(\mathcal{O})$ -equivariant perverse sheaves on the affine Grassmannian, and is called the Satake category.

• By t -exactness of $\overline{\mathcal{D}}_{\mathbf{I}}$, it extends to a functor

$$S : (\text{Rep } G^{\vee})^{\otimes \mathbf{I}} \longrightarrow \text{Shv}(\text{Hecke}_{\mathbf{I}}^{\text{loc}}),$$

which we call the **Satake functor**.

We have the following correspondence diagram:



Finally, we may construct the **Hecke functor**:

$$\begin{aligned}
 H_V : \text{Shv}(\text{Bun}_G) &\longrightarrow \text{Shv}(\text{Bun}_G \times X^I) \\
 \mathcal{F} &\longmapsto (\vec{h} \times s)_* (\vec{h} \times r)^* (\mathcal{F} \boxtimes S_V)
 \end{aligned}$$

(III) Restricting Hecke Functor to $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$.

Our main goal is to prove the following theorem typically attributed to Nadler & Yun: (Remark: Ginzburg in fact proved this statement in §6.6 of his paper on "Perverse sheaves on a loop group & Langlands duality"!)
 $\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \rightarrow \text{Shv}_{\text{Nilp} \times \text{SOS}}(\text{Bun}_G \times X^I)$

Thm The Hecke functor restricts to give a functor

$$H_V : \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \longrightarrow \text{Shv}_{\text{Nilp} \times \text{SOS}}(\text{Bun}_G \times X^I)$$

pf By associativity of $H : (\text{Rep } G^V)^{\otimes I} \otimes \text{Shv}(\text{Bun}_G) \rightarrow \text{Shv}(\text{Bun}_G \times X^I)$

we may assume $I = \text{pt}$.
Next, recall the two characteristics of singular support with respect to smooth pullback $(\vec{h} \times r)$ & proper pushforward $(\vec{h} \times s)$:

Given $F : U \rightarrow V$, consider

$$T^*U \xleftarrow{dF} T^*V \times_V U \xrightarrow{(\text{df})_*} T^*V$$

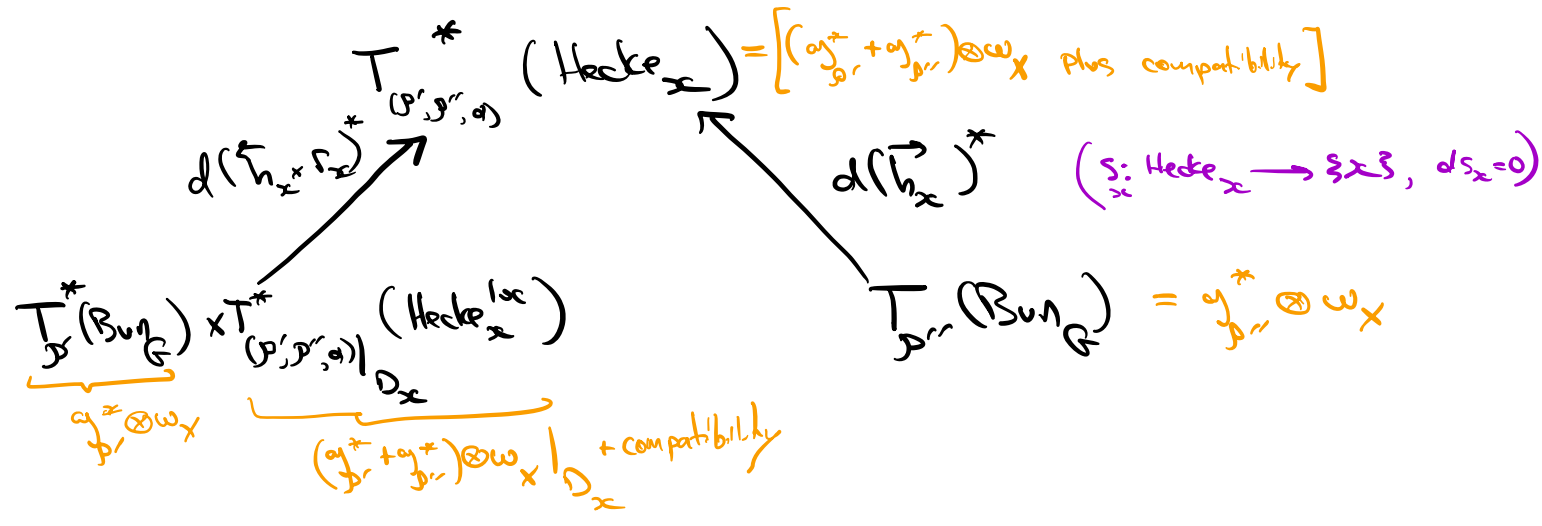
$$\begin{aligned}
 \text{i.e., } (dF)_\downarrow &:= f_{\sharp} \circ dF^{-1} \\
 (dF)_\uparrow &:= dF \circ f_{\sharp}^{-1}
 \end{aligned}$$

(dF)* WANT $\mathcal{N} \times \{0\}$

$$SS(H_V(\mathcal{F})) \cong d(\vec{h} \times s)_* d(\vec{h} \times r)^* (SS(\mathcal{F} \boxtimes S_V)) \cong T^*(\text{Bun}_G \times X)$$

GOAL! Now, consider $(\xi', \xi_H) \in SS(\mathcal{F} \boxtimes S_V) = \mathcal{N} \times SS(S_V)$
 and $(\xi'', \xi_X) \in SS(H_V(\mathcal{F}))$ such that
 (+) $d(\vec{h} \times r)^* (\xi', \xi_H) = d(\vec{h} \times s)^* (\xi'', \xi_X)$.
WANT TO SHOW ξ'' also nilpotent and $\xi_X = 0$.

Lets consider the fiber over $x \in X$ of the differential of the Hecke correspondence:



Recall:

$$T_{p'}^*(\text{Bun}_G) = H^1(X, g_{p'}^*) = \Gamma(X, g_{p'}^* \otimes \omega_X)$$

$$T_{(p', p'', \vartheta)}^*(\text{Hecke}_x) = H^1(X, K)^*$$

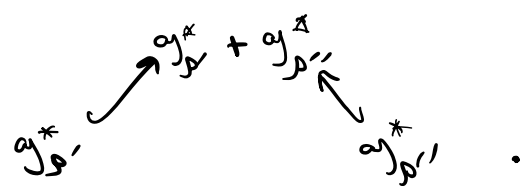
where $K = \text{Con}(g_{p'} \oplus g_{p''} \rightarrow i_* i^* g_{p'} \simeq i_* i^* g_{p''}), i: X \setminus x \hookrightarrow X$.

$\uparrow d\tau_x^*$

$$T_{(p', p'', \vartheta)}^*(\text{Hecke}_x^{\text{loc}}) = H^1(D_x, K)^*$$

(++) explicitly $\left\{ \begin{aligned} & (\xi'_{\text{loc}}, \xi''_{\text{loc}}) \in \Gamma(X, (g_{p'}^* \oplus g_{p''}^*) \otimes \omega_X) \text{ such that} \\ & \vartheta(\xi'_{\text{loc}}|_{D_x^0}) = \xi''_{\text{loc}}|_{D_x^0} \end{aligned} \right\}$

Under these identifications, $d\tilde{h}_x^*$, $d\tilde{h}_x^*$ are just reduced by the inclusions



The fiber of (t) over $x \in X \Rightarrow$

$$d\tilde{h}_x^*(\xi') + (d\tilde{h}_x^*)^*(\xi_{H_x}) = d\tilde{h}_x^*(\xi'') \in T^*(\text{Hecke}_x)$$

$\xrightarrow{\in T^*(\text{Hecke}_x^{\text{loc}})}$

Thus ξ' nilpotent $\Leftrightarrow \xi' \Big|_{\mathcal{O}_x^0}$ nilp.

$\Leftrightarrow \xi'' \Big|_{\mathcal{O}_x^0}$ nilp

$\Leftrightarrow \xi''$ nilp.

The compatibility constraint of cocovers in Hecke stack ensures one cocover is nilp \Leftrightarrow the other is nilp

Remains to show $\xi_X = 0$.

Sketch: The SES admits a canonical splitting:

$$0 \rightarrow T_x^* X \xrightarrow{\xi_H} T_{(-)}^*(\text{Hecke}_X) \rightarrow T(\text{Hecke}) \rightarrow 0$$

Thus, suffices: If $\xi_H \in T^*(\text{Hecke}_x)$ is in $SS(S_v)$ & $\xi_{H_x} = (\xi'_{\text{loc}}, \xi''_{\text{loc}}) \in T^*(\text{Hecke}_x^{\text{loc}})$ is nilpotent, then image of ξ_H under splitting is 0.

This is done by an explicit calculation, by working with $X = \mathbb{A}^1$, $x=0$ (since statement is local), using

$$(p', p'', a) \in \text{Hecke}_0^{\text{loc}} = G(\mathbb{O}) \backslash G(K) / G(\mathbb{O}) \rightsquigarrow g \in G(K) + \text{data.}$$

Then, use Cartan decomp to reduce to $g = z^{-1}$, where $y_K = \mathcal{O}(K)$.

Then, the canonical splitting takes the form:

$$T^* \text{Herkloc} = \int_0^1 \text{Ad}_{t^{\lambda}}(\mathfrak{y}_0^*) dt = \int_0^1 \mathfrak{y}_k^* dt$$

spl. Aug

$$T_0^* A' = \mathbb{C}$$

$$\begin{array}{c} \cong \\ \downarrow \\ \text{Res}_{t=0} \langle \lambda t^{-1}, \mathfrak{z} \rangle \\ \text{dlog}(0) \end{array}$$

Reduced to showing: $\exists F \bar{\mathfrak{z}}_H := \int_H \text{mod } t \in \mathfrak{y}^*$ is nilpotent

then $\langle \lambda, \bar{\mathfrak{z}}_H \rangle (= \text{Res}_{t=0} \langle \lambda t^{-1}, \mathfrak{z}_H \rangle)$ is zero. Exercise! \blacksquare

Finally, the last ingredient we need is the following:

Suppose N, N' are Lagrangian substacks of T^*X, T^*X' . Then

$$\text{Shv}_{N \times N'}(X \times Y) = \text{Shv}_N(X) \otimes \text{Shv}_{N'}(Y)$$

(this kind of generality only holds in Betti setting)

(IV) Construction of the global Hodge action in general

Let us explain the construction in general (Beilinson, de Rham, etc).

First, we must replace $\text{Loc Sys}_{G^v}(X)$ with $\text{Loc Sys}_{G^v}^{\text{rest}}(X)$,

where, recall from Nick's talk

$$\text{Loc Sys}_{G^v}^{\text{rest}}(X)(S) = \left\{ \begin{array}{l} \text{right t-exact } \otimes\text{-functors} \\ \text{Rep } G^v \rightarrow \text{QLisse}(X) \otimes \text{QCoh}(S) \end{array} \right\}, \quad \text{where}$$

• $\text{QLisse}^{\text{Betti}}(X) \subseteq \text{Shv}_{\text{l.c.}}(X(\mathbb{C}))$ is complexes whose cohom.

is given by ind-finite-dim. $\mathbb{Z}_\ell(X(\mathbb{C}))$ -reps,

• $\text{QLisse}^{\text{ét}}(X) \subseteq \text{Shv}^{\text{ét}}(X)$ consists of complexes of Ind-constructible l-adic sheaves w/ ind-locally constant cohomology sheaves,

• $\mathcal{Q}Lisse^{\text{nr}}(X) \subseteq D\text{-mod}(X)$ consists of incl- (regular local systems).

Main Thm (8.3.7 AGKRRV) $X = \text{smooth, proper curve}$.

Then $(\text{Rep } G^V)^{\otimes X - \text{Lisse}}$:= cotHom $(\text{Rep } G^V, \mathcal{Q}Lisse(X))$

is equivalent to $\mathcal{Q}Coh(\text{Loc Sys}_{G^V}^{\text{restr}})$.

Remark (1) Given $\mathcal{Q} \in \text{DGCat}^{\text{Sym Mon}}$ which is dualizable, & $\mathcal{D} \in \text{DGCat}^{\text{Sym Mon}}$,
define cotHom $(\mathcal{C}, \mathcal{Q}) \in \text{DGCat}^{\text{Sym Mon}}$ using universal property:

$$\text{Fun}^{\otimes}(\text{cotHom}(\mathcal{C}, \mathcal{Q}), \mathcal{D}) := \text{Fun}^{\otimes}(\mathcal{C}, \mathcal{D} \otimes \mathcal{Q}).$$

$$(2) \text{cotHom}(\text{Rep } G^V, \mathcal{L}S(X)) = (\text{Rep } G^V)^{\otimes X}$$

(3) Even in Beilinson setting we proved, additional work is needed to go from Beilinson to restricted Beilinson.

(Note, we're now working with Incl-constructible sheaves instead of all sheaves!)

Main Thm 2 [Nadler - Yun analogue] There is an

action

$$(\text{Rep } G^V)^{\otimes \mathbb{I}} \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \rightarrow \text{Shv}_{\text{Nilp} \times \text{fob}}(\text{Bun}_G \times X^{\mathbb{I}})$$

where $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ is interpreted differently, depending on the sheaf theory. E.g., for étale, we use Sasha's definition.

The final ingredient is the decomposition

$$\text{Shv}_{\text{Nilp} \times \text{fob}}(\text{Bun}_G \times X^{\mathbb{I}}) = \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \mathcal{Q}Lisse(X)^{\otimes \mathbb{I}}.$$

Con There exists an adon $\mathcal{Q}\text{Coh}(\text{LocSys}_{G^v}^{\text{rest}}) \rightsquigarrow \text{Shv}_{\text{v:tp}}(\text{Bun}_G)$